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A POINTWISE CHARACTERIZATION OF FUNCTIONS OF BOUNDED VARIATION ON METRIC SPACES

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Abstract. We give a new characterization of the space of functions of bounded variation in terms of a pointwise inequality connected to the maximal function of a measure. The characterization is new even in Euclidean spaces and it holds also in general metric spaces.

1. Introduction

There are several equivalent definitions for functions of bounded variation in Euclidean spaces. Two of the most common, which we recall below, cannot be generalized to metric spaces because they make use of smooth functions and (weak) derivatives. An integrable function u is a function of bounded variation, $u \in BV(\mathbb{R}^n)$, if

$$\|Du\|(\mathbb{R}^n) = \sup\Big\{ \int_{\mathbb{R}^n} u \operatorname{div} \varphi \, dx : \varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n), \|\varphi\|_\infty \le 1 \Big\} < \infty,$$

or, equivalently, if there exist real finite measures $\mu_1, \dots, \mu_n$ such that

$$\int_{\mathbb{R}^n} u D_i \varphi \, dx = - \int_{\mathbb{R}^n} \varphi \, d\mu_i \text{ for all } \varphi \in C_c^1(\mathbb{R}^n), \; i = 1, \dots, n,$$

that is, the weak gradient $Du = \mu$ of u is an $\mathbb{R}^n$-valued measure with finite total variation $|Du|(\mathbb{R}^n)$. The above definitions are equivalent to the following one, based on a relaxation procedure using Lipschitz functions. A function $u \in L^1(\mathbb{R}^n)$ belongs to $BV(\mathbb{R}^n)$, if

$$L(u) = \inf\Big\{ \liminf_{i\to\infty} \int_{\mathbb{R}^n} |\nabla u_i| \, dx : u_i \in \operatorname{Lip}(\mathbb{R}^n), \, u_i \to u \text{ in } L^1(\mathbb{R}^n) \Big\} < \infty. \tag{1.1}$$

The definition given by (1.1) has been generalized to a metric measure space by using the local Lipschitz constant (2.6) in the place of the gradient by Ambrosio in [1] and Miranda in [19]. This definition together with the doubling property of the measure and the validity of a $(1,1)$-Poincaré inequality provides a rich theory of functions of bounded variation in metric spaces, see for example [1], [2], [4], [5], [13], [16], [17], [18], [19]. Properties of BV functions in $\mathbb{R}^n$ can be studied from the monographs [3] (contains a historical overview in Section 3.12), [6], [7], [8], [21].

In this paper, motivated by the characterization of the Sobolev space $W^{1,1}(\mathbb{R}^n)$ given by Hajłasz in [11], we give a new characterization of functions of bounded variation in metric spaces using a pointwise estimate. The characterization is new even in the classical

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setting. For the notation and definitions used in the introduction and throughout the paper, see Section 2.

Before giving the characterization, we recall the inequality behind the Sobolev spaces $M^{1,p}(X)$, where $X = (X, \mathrm{d}, \mu)$ is a metric measure space. For $1 < p < \infty$, the function $u \in L^p(\mathbb{R}^n)$ belongs to $W^{1,p}(\mathbb{R}^n)$ if and only if there is a function $0 \le g \in L^p(\mathbb{R}^n)$ such that the pointwise inequality

$$|u(x) - u(y)| \le |x - y|\big(g(x) + g(y)\big) \tag{1.2}$$

holds for almost all $x, y \in \mathbb{R}^n$, see [9]. The validity of (1.2) for $u \in W^{1,p}(\mathbb{R}^n)$ follows from the inequality

$$|u(x) - u(y)| \le C(n)|x - y|\big[\mathcal{M}_{2|x-y|}\,|\nabla u|(x) + \mathcal{M}_{2|x-y|}\,|\nabla u|(y)\big] \tag{1.3}$$

for almost all $x, y \in \mathbb{R}^n$, which holds for all $1 \le p < \infty$, and the L^p-boundedness of the Hardy-Littlewood maximal operator $\mathcal{M}$ for $p > 1$, see for example [9]. The boundedness is essential; for a function $u \in W^{1,1}(\mathbb{R}^n)$ there is not necessarily any integrable function g such that inequality (1.2) holds, see [11]. In [11], Hajłasz gave the following characterization of $W^{1,1}(\mathbb{R}^n)$ using a pointwise estimate with maximal functions on its right-hand side.

Theorem 1.1 ([11, Theorem 4])**.** *Let $u \in L^1(\mathbb{R}^n)$. Then $u \in W^{1,1}(\mathbb{R}^n)$ if and only if there exists a function $0 \le g \in L^1(\mathbb{R}^n)$ and a constant $\sigma \ge 1$ such that the pointwise inequality*

$$|u(x) - u(y)| \le |x - y|\big[\mathcal{M}_{\sigma|x-y|}\,g(x) + \mathcal{M}_{\sigma|x-y|}\,g(y)\big] \tag{1.4}$$

holds for almost all $x, y \in \mathbb{R}^n$.

In the metric setting, we can characterize Newtonian functions by a similar pointwise inequality, provided the space supports the $(1,p)$-Poincaré inequality (2.7). Recall that Newtonian spaces are a generalization of Sobolev spaces to metric spaces using upper gradients, see [20]. For any $p > 0$, a $(1,p)$-Poincaré inequality for a pair $u \in L^1_{\mathrm{loc}}(X)$ and a measurable function $g \ge 0$ implies, using a standard chaining argument, a pointwise inequality of the same type as (1.4),

$$|u(x) - u(y)| \le C\,\mathrm{d}(x,y)\Big[\big(\mathcal{M}_{2\tau\,\mathrm{d}(x,y)}\,g^p(x)\big)^{1/p} + \big(\mathcal{M}_{2\tau\,\mathrm{d}(x,y)}\,g^p(y)\big)^{1/p}\Big] \tag{1.5}$$

for μ-almost all $x, y \in X$, see [12, Theorem 3.2]. A converse holds when $p > s/(s+1)$, where s is the doubling dimension: if the pair $u, g \in L^p(X)$ satisfies the pointwise inequality (1.5), it also satisfies a $(1,p)$-Poincaré inequality, see [10, Theorem 9.5]. Furthermore, if $p \ge 1$ and X is complete, then according to [10, Theorem 11.2], it follows that u belongs to the Newtonian space $N^{1,p}(X)$. Thus the pointwise inequality (1.5) for $u, g \in L^p(X)$ characterizes the space $N^{1,p}(X)$ for any $1 \le p < \infty$.

For BV functions, Poincaré inequality (2.8) and the same proof as in [12, Theorem 3.2] give a similar estimate as (1.5) for the oscillation of a function. Namely, if $u \in BV(X)$, then for μ-almost all $x, y \in X$,

$$|u(x) - u(y)| \le C\,\mathrm{d}(x,y)\big[\,\mathcal{M}_{2\tau\,\mathrm{d}(x,y),\|Du\|}(x) + \mathcal{M}_{2\tau\,\mathrm{d}(x,y),\|Du\|}(y)\big], \tag{1.6}$$

where the constant $C > 0$ depends only on the doubling constant c_d and on the constants of the Poincaré inequality. Here $\mathcal{M}_{2\tau d,\|Du\|}$ is the restricted maximal function (2.4) of the measure $\|Du\|$. In Theorem 3.2, we show that a similar pointwise inequality (3.1) with a maximal function of a measure implies a Poincaré type inequality (3.2) with the same

measure on the right hand side. This, together with Theorem 1.2 by Miranda, shows that u is a function of bounded variation.

Theorem 1.2 ([19, Theorem 3.8]). *Let X be a complete, doubling metric measure space that supports a $(1,1)$-Poincaré inequality. Let $u \in L^1(X)$. Then $u \in BV(X)$ if and only if there exist constants $C_1 > 0$ and $\eta > 0$ and a positive, finite measure ν such that*

$$\int_B |u - u_B|\, d\mu \le C_1 r \nu(\eta B)$$

for each ball $B(x,r)$. Moreover, $\|Du\| \le C\nu$, with $C = C(C_1, c_d, \eta)$.

We obtain our characterization by combining (1.6), Theorem 3.2, and Theorem 1.2. Although the characterization is new even in Euclidean spaces, we formulate it only in the general metric space setting.

Theorem 1.3. *Let X be a complete, doubling metric measure space that supports a $(1,1)$-Poincaré inequality. Let $u \in L^1(X)$. Then $u \in BV(X)$ if and only if there exists a positive, finite measure ν and constants $\sigma \ge 1$ and $C_0 > 0$ such that the inequality*

$$|u(x) - u(y)| \le C_0\, \mathrm{d}(x,y) \big[\mathcal{M}_{\sigma\, \mathrm{d}(x,y),\nu}(x) + \mathcal{M}_{\sigma\, \mathrm{d}(x,y),\nu}(y) \big]$$

holds for μ-almost all $x, y \in X$. Moreover, $\|Du\| \le C\nu$, where C only depends on C_0, σ, the doubling constant of the measure, and the constants in the $(1,1)$-Poincaré inequality.

2. Notation and preliminaries

We assume that $X = (X, \mathrm{d}, \mu)$ is a metric measure space equipped with a metric d and a Borel regular, *doubling* outer measure μ. The doubling property means that there is a fixed constant $c_d > 0$, called the *doubling constant* of μ, such that

$$\mu(2B) \le c_d \mu(B) \tag{2.1}$$

for every ball $B = B(x,r) = \{y \in X : \mathrm{d}(y,x) < r\}$. Here $tB = B(x,tr)$. We assume that the measure of every open set is positive and that the measure of each bounded set is finite. The doubling condition gives an upper bound for the dimension of X. By this we mean that there is a constant $C = C(c_d) > 0$ and an exponent $s \ge 0$ such that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \ge C\Big(\frac{r}{R}\Big)^s \tag{2.2}$$

whenever $0 < r \le R < \operatorname{diam}(X)$, $x \in X$, and $y \in B(x,R)$. Inequality (2.2) holds certainly with $s = \log_2 c_d$ (but it may hold for some smaller exponents as well). We call s the doubling dimension of X.

We also assume that X is complete; recall that a metric space with a doubling measure is complete if and only if the space is proper, that is, closed and bounded sets are compact. When we say that an inequality such as (1.5) holds for μ-almost all $x, y \in X$, we mean that there is a set $E \subset X$ such that the property holds for all $x, y \in X \setminus E$, and $\mu(E) = 0$.

The restricted Hardy-Littlewood maximal function of a locally integrable function u is

$$\mathcal{M}_R\, u(x) = \sup_{0<r\le R} \fint_{B(x,r)} |u(y)|\, d\mu(y), \tag{2.3}$$

where $u_B = \fint_B u\, d\mu = \mu(B)^{-1} \int_B u\, d\mu$ is the integral average of u over B. For $R = \infty$, $\mathcal{M}_\infty\, u$ is the usual Hardy-Littlewood maximal function $\mathcal{M}\, u$.

Similarly, the restricted maximal function of a positive, finite measure ν is

$$\mathcal{M}_{R,\nu}(x) = \sup_{0<r\le R} \frac{\nu(B(x,r))}{\mu(B(x,r))}. \tag{2.4}$$

For $R=\infty$, we write $\mathcal{M}_\nu$.

A curve is a rectifiable continuous mapping from a compact interval to X. A nonnegative Borel function g on X is an upper gradient of an extended real valued function u on X, if for all curves γ in X, we have

$$|u(x)-u(y)| \le \int_\gamma g\,ds, \tag{2.5}$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma g\,ds=\infty$ otherwise. Here x and y are the end points of γ. If g is a nonnegative measurable function on X and (2.5) holds for almost every curve with respect to the 1-modulus, then g is a 1-weak upper gradient of u. For the concept of modulus in metric spaces, see [15]. A natural upper gradient for a Lipschitz function u is the local Lipschitz constant

$$\operatorname{Lip} u(x) = \liminf_{r\to 0} \sup_{y\in B(x,r)} \frac{|u(x)-u(y)|}{\mathrm{d}(x,y)}. \tag{2.6}$$

Next we recall the definition of functions of bounded variation on metric spaces, given by Miranda in [19].

Definition 2.1. For $u\in L^1_{\mathrm{loc}}(X)$, we define

$$\|Du\|(X) = \inf\Big\{ \liminf_{i\to\infty} \int_X \operatorname{Lip} u_i\,d\mu : u_i\in \operatorname{Lip}_{\mathrm{loc}}(X), u_i\to u \text{ in } L^1_{\mathrm{loc}}(X)\Big\},$$

and we say that a function $u\in L^1(X)$ is of bounded variation, $u\in BV(X)$, if $\|Du\|(X)<\infty$. Note that replacing the Lipschitz constants with 1-weak upper gradients in the definition yields the same space.

We say that X supports a (weak) $(1,p)$-Poincaré inequality, $0<p<\infty$, if there exist constants $c_P>0$ and $\tau\ge 1$ such that for all balls $B=B(x,r)$, all locally integrable functions u, and all p-weak upper gradients g of u, we have

$$\fint_B |u-u_B|\,d\mu \le c_P r\Big(\fint_{\tau B} g^p\,d\mu\Big)^{1/p}. \tag{2.7}$$

If the space supports a $(1,1)$-Poincaré inequality, then for every $u\in BV(X)$ we have

$$\fint_B |u-u_B|\,d\mu \le c_P r\frac{\|Du\|(\tau B)}{\mu(\tau B)}, \tag{2.8}$$

where the constant c_P and the dilation factor τ are the same as in (2.7). Inequality (2.8) follows easily by using (2.7) for approximating Lipschitz functions in the definition of $BV(X)$.

The characteristic function of a set $E\subset X$ is χ_E. Both the Euclidean distance and the Lebesgue measure in $\mathbb{R}^n$ are denoted by $|\cdot|$. In general, C will denote a positive constant whose value is not necessarily the same at each occurrence.

3. Pointwise estimate and Poincaré inequality

We begin with a geometric lemma. Recall that X is *a geodesic space* if every two points $x, y \in X$ can be joined by a curve whose length is equal to $\mathrm{d}(x, y)$.

Lemma 3.1. *Let X be a geodesic metric space. If $B(x_0, R)$ is a ball, $x \in B(x_0, R)$, and $0 < r \le 2R$, then there is a ball of radius $r/2$ in $B(x, r) \cap B(x_0, R)$.*

Proof. If $\mathrm{d}(x, x_0) \ge r/2$, then the assumption that X is geodesic implies that there is a point z such that $\mathrm{d}(z, x) = r/2$ and $\mathrm{d}(z, x_0) = \mathrm{d}(x, x_0) - r/2$, and hence $B(z, r/2) \subset B(x, r) \cap B(x_0, R)$.

On the other hand, if $\mathrm{d}(x, x_0) < r/2$, then $B(x_0, r/2) \subset B(x, r) \cap B(x_0, R)$. □

The idea of the proof of the next theorem is from [10] and [11].

Theorem 3.2. *Let X be a complete, doubling metric measure space that supports a $(1,1)$-Poincaré inequality. Let $u \in L^1_{loc}(X)$, and let ν be a positive, finite measure. If there are constants $\sigma \ge 1$ and $C_0 > 0$ such that the inequality*

$$|u(x) - u(y)| \le C_0 \,\mathrm{d}(x, y) \big[\mathcal{M}_{\sigma \,\mathrm{d}(x,y),\nu}(x) + \mathcal{M}_{\sigma \,\mathrm{d}(x,y),\nu}(y) \big] \tag{3.1}$$

holds for μ-almost all $x, y \in X$, then

$$\int_B |u - u_B| \, d\mu \le C r \nu(\eta B) \tag{3.2}$$

for each ball $B = B(x, r)$. The constants C and η depend only on C_0, σ, the doubling constant of the measure, and the constants in the $(1,1)$-Poincaré inequality.

Proof. Since X supports a $(1,1)$-Poincaré inequality, X is quasiconvex. This means that there is a constant $C \ge 1$, depending only on the doubling constant c_d and the constants in the Poincaré inequality, such that every two points $x, y \in X$ can be connected by a curve γ satisfying $\ell(\gamma) \le C \,\mathrm{d}(x, y)$, where $\ell(\gamma)$ is the length of γ, see [12, Proposition 4.4]. This implies that X endowed with the length metric $\rho(x, y) = \inf \ell(\gamma)$, where the infimum is taken over all curves connecting x and y, is bi-Lipschitz homeomorphic to X. Since the doubling property, the $(1,1)$-Poincaré inequality and inequalities (3.1) and (3.2) are invariant under bi-Lipschitz homeomorphisms (cf. [14, Chapter 9]), we may replace the metric d by ρ and work with the new space (X, ρ, μ). Therefore, throughout the proof we will assume that d is the length metric. Since X is proper, such a metric has the property that every two points $x, y \in X$ can be connected by a geodesic, that is, a curve whose length equals $\mathrm{d}(x, y)$, see [10, Theorem 3.9].

Let $B = B(x_0, R)$ be a ball. We begin the proof by checking what we can assume from u and ν. Since neither inequality (3.1) nor inequality (3.2) change if a constant is added to u, we may assume that $\operatorname{ess\,inf}_E |u| = 0$ for a set $E \subset B$ with $\mu(E) > 0$. We will choose the set E later.

We define $\tau = 3\sigma$, and $\lambda = \nu|_{\tau B}$. The pointwise estimate (3.1) implies that,

$$|u(x) - u(y)| \le C_0 \,\mathrm{d}(x, y) \big[\mathcal{M}_\lambda(x) + \mathcal{M}_\lambda(y) \big] \tag{3.3}$$

for almost all $x, y \in B$. We may assume that (3.3) holds for all $x, y \in B$ because inequality (3.2) with B replaced by $B \setminus F$, where $\mu(F) = 0$, implies (3.2) with B.

Moreover, we may assume that $\lambda(\tau B) > 0$, since otherwise u is constant in B, and inequality (3.2) follows. For each $k \in \mathbb{Z}$, we define

$$E_k = \{x \in B : \mathcal{M}_\lambda(x) \le 2^k\} \quad \text{and} \quad a_k = \sup_{E_k} |u(x)|.$$

Then $E_{k-1} \subset E_k$ and $a_{k-1} \le a_k$ for each k, and

$$\int_B |u - u_B|\, d\mu \le 2 \int_B |u|\, d\mu \le 2 \sum_{k=-\infty}^{\infty} a_k \mu(E_k \setminus E_{k-1}). \tag{3.4}$$

We will obtain an upper bound for the right hand side of this inequality by estimating the values of a_k. By the pointwise estimate (3.3), the function u is $C_0 2^{k+1}$-Lipschitz in E_k. Hence, for each $x \in E_k$ and $y \in E_{k-1}$, we have

$$|u(x)| \le |u(x) - u(y)| + |u(y)| \le C_0 2^{k+1} \,\mathrm{d}(x, y) + a_{k-1}. \tag{3.5}$$

Our next goal is to find for each $x \in E_k$ a point $y \in E_{k-1}$ such that the distance from y to x is sufficiently small. Fix $x \in E_k$. By Lemma 3.1, $B(x, r) \cap B$ contains a ball $\tilde{B}$ of radius $r/2$ if $0 < r \le 2R$, and hence, by the doubling property of μ,

$$\mu(B(x, r) \cap B) \ge \mu(\tilde{B}) \ge \mu(B) c_d^{-2} \Big(\frac{r}{2R}\Big)^s, \tag{3.6}$$

where $s = \log_2 c_d$. Since s can always be replaced by a larger number, we can assume that $s > 1$. We also have the weak type estimate

$$\mu(B \setminus E_{k-1}) = \mu\big(\{x \in B : \mathcal{M}_\lambda(x) > 2^{k-1}\}\big) < \frac{C}{2^{k-1}} \lambda(\tau B). \tag{3.7}$$

Thus, in order to obtain the inequality $\mu(B(x, r) \cap B) > \mu(B \setminus E_{k-1})$, it is sufficient to require that

$$\mu(B) c_d^{-2} \Big(\frac{r}{2R}\Big)^s \ge \frac{C}{2^{k-1}} \lambda(\tau B),$$

that is,

$$r \ge 2R \Big(\frac{C\lambda(\tau B)}{2^{k-1}\mu(B)}\Big)^{1/s}.$$

Let us thus define for each $k \in \mathbb{Z}$

$$r_k := 2R \Big(\frac{C\lambda(\tau B)}{2^{k-1}\mu(B)}\Big)^{1/s}.$$

Now, since $\mu(B(x, r_k) \cap B) > \mu(B \setminus E_{k-1})$, there is a $y \in B(x, r_k) \cap E_{k-1}$. The definition of r_k and (3.5) then imply that

$$a_k \le a_{k-1} + C_0 2^{k+1} r_k = a_{k-1} + C_0 2^{k+2} R \Big(\frac{C\lambda(\tau B)}{2^{k-1}\mu(B)}\Big)^{1/s}.$$

Iterating the above estimate starting from some $k_0 \in \mathbb{Z}$, we get

$$\begin{aligned} a_k &\le a_{k_0} + \sum_{i=k_0+1}^{k} CR2^{i(1-1/s)}\Big(\frac{\lambda(\tau B)}{\mu(B)}\Big)^{1/s} \\ &\le a_{k_0} + CR\Big(\frac{\lambda(\tau B)}{\mu(B)}\Big)^{1/s} \sum_{i=-\infty}^{k} 2^{i(1-1/s)} \\ &\le a_{k_0} + CR\Big(\frac{\lambda(\tau B)}{\mu(B)}\Big)^{1/s} 2^{k(1-1/s)} \end{aligned} \tag{3.8}$$

for each $k > k_0$. Note that $a_k \le a_{k_0}$ for each $k \le k_0$. Now, as mentioned before equation (3.6), we must require that $r_{k_0+1} \le 2R$, that is,

$$2R\Big(\frac{C\lambda(\tau B)}{2^{k_0}\mu(B)}\Big)^{1/s} \le 2R,$$

or equivalently

$$\frac{C\lambda(\tau B)}{2^{k_0}} \le \mu(B).$$

Let $k_0 \in \mathbb{Z}$ be the smallest integer for which the above inequality holds. We then have

$$2^{k_0} \ge \frac{C\lambda(\tau B)}{\mu(B)}$$

and

$$2^{k_0-1} < \frac{C\lambda(\tau B)}{\mu(B)}.$$

By the doubling property of μ we thus have for some constant C

$$\frac{1}{C}\frac{\lambda(\tau B)}{\mu(\tau B)} \le 2^{k_0} \le C\frac{\lambda(\tau B)}{\mu(\tau B)}. \tag{3.9}$$

We select the set E discussed in the beginning of the proof to be E_{k_0}. Note that $\mu(E_{k_0}) > 0$ because $\mu(B(x, r_{k_0+1}) \cap B) > \mu(B \setminus E_{k_0})$ for any $x \in B$. As mentioned in the beginning of the proof, we can now assume that $\operatorname{ess\,inf}_{E_{k_0}} |u| = 0$. Then, by the $C_0 2^{k_0+1}$-Lipschitz continuity of u in E_{k_0}, and (3.9) we have the following estimate for a_{k_0}:

$$a_{k_0} = \sup_{E_{k_0}} |u| \le C_0 2^{k_0+1} \cdot 2R \le CR\frac{\lambda(\tau B)}{\mu(\tau B)}. \tag{3.10}$$

By writing $A_k = E_k \setminus E_{k-1}$ and using (3.4) and (3.8), we have

$$\begin{aligned} \frac{1}{2}\int_B |u - u_B|\,d\mu &\le \sum_{k=-\infty}^{\infty} a_k\mu(A_k) \\ &\le \sum_{k=-\infty}^{k_0} a_{k_0}\mu(A_k) + \sum_{k=k_0+1}^{\infty} \Big(a_{k_0} + CR\Big(\frac{\lambda(\tau B)}{\mu(B)}\Big)^{1/s} 2^{k(1-1/s)}\Big)\mu(A_k) \\ &\le \sum_{k=-\infty}^{\infty} a_{k_0}\mu(A_k) + CR\Big(\frac{\lambda(\tau B)}{\mu(B)}\Big)^{1/s} \sum_{k=k_0+1}^{\infty} 2^{k(1-1/s)}\mu(B \setminus E_{k-1}), \end{aligned} \tag{3.11}$$

where, by (3.10) and the doubling property of μ,

$$\sum_{k=-\infty}^{\infty} a_{k_0}\mu(A_k) \le CR\frac{\lambda(\tau B)}{\mu(\tau B)}\mu(B) \le CR\lambda(\tau B).$$

Moreover, we estimate the last sum of (3.11) by using the weak type estimate (3.7) and (3.9):

$$\begin{aligned}\sum_{k=k_0+1}^{\infty} 2^{k(1-1/s)}\mu(B\setminus E_{k-1}) &\le \sum_{k=k_0+1}^{\infty} 2^{k(1-1/s)}\frac{C\lambda(\tau B)}{2^{k-1}}\\ &\le C\lambda(\tau B)\sum_{k=k_0+1}^{\infty} 2^{-k/s}\\ &\le C\lambda(\tau B)2^{-k_0/s}\\ &\le C\lambda(\tau B)\Big(\frac{\mu(\tau B)}{\lambda(\tau B)}\Big)^{1/s}.\end{aligned}$$

Finally, using the doubling property of μ, we get

$$\frac{1}{2}\int_B |u-u_B|\,d\mu \le CR\lambda(\tau B).$$

Hence the claim follows — we only have to note that the constants in the final form of the Poincaré type inequality will possibly be altered by the swaps between the metrics discussed in the beginning of the proof. □

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